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Fourth-order exponential finite difference methods for boundary value problems of convective diffusion type

A. C. Radhakrishna Pillai*[,1]

Aerodynamics Division, Aeronautical Development Establishment, Bangalore, India

SUMMARY

Methods based on exponential finite difference approximations of h^4 accuracy are developed to solve one and two-dimensional convection–diffusion type differential equations with constant and variable convection coefficients. In the one-dimensional case, the numerical scheme developed uses three points. For the two-dimensional case, even though nine points are used, the successive line overrelaxation approach with alternating direction implicit procedure enables us to deal with tri-diagonal systems. The methods are applied on a number of linear and non-linear problems, mostly with large first derivative terms, in particular, fluid flow problems with boundary layers. Better accuracy is obtained in all the problems, compared with the available results in the literature. Application of an exponential scheme with a non-uniform mesh is also illustrated. The h^4 accuracy of the schemes is also computationally demonstrated. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: boundary layer; convection–diffusion; exponential scheme; finite differencing; non-uniform mesh

1. INTRODUCTION

The convective diffusion equation is of primary importance in many physical systems, especially those involving fluid flow. Numerical solutions of such model equations with general finite differencing are known to give spatially oscillatory results. Several upwinding schemes have been developed over the years to obtain reasonably accurate solutions and to eliminate the spurious oscillations. Most of the numerical schemes use centred differencing for the second-order derivative diffusion term and some kind of upwind differencing for the convection terms. The stability of the first-order upwind scheme is good, but this has a strong diffusive effect and suffers because of low accuracy. The second-order upwind scheme has undesirable propagation of errors and stability problems.

* Correspondence to: Aerodynamics Division, Aeronautical Development Establishment, C.V. Raman Nagar, Bangalore 560093, India. Tel.: +91 80 5281480; fax: +91 80 5283188.
[1] E-mail: acrp@ade.ernet.in

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General finite difference approximation for the convection–diffusion problem was discussed by Anderson *et al.* [1]. A review of the upwind schemes, including the third-order upwind scheme of Leonard [2], was given by Patel *et al.* [3] and Leschziner [4]. Kawamura *et al.* [5] developed a third-order accurate upwind scheme by modifying the conventional second-order upwind scheme. This resulted in a complex five-point approximation for the convective term, which makes it difficult to approximate the points adjacent to the boundaries. Dekema and Schultz [6] developed higher-order methods for the convection–diffusion equation using the Taylor series expansion, where the higher-order derivatives were replaced by using the differential equation. Application of these methods to the two-dimensional non-linear problems appears to be quite difficult due to the complexity involved in the calculation of the parameters. The concept of a tensor viscosity method for differencing the convective terms for hyperbolic flows has been described by Dukowicz and Ramshaw [7]. Abarbanel and Kumar [8] employed the original equation in two-dimensions to obtain a nine-point spatially $O(h^4)$, temporally $O(h^2)$ scheme for the Euler equations; they also extended it to three dimensions.

Finite difference approximations involving exponential coefficients and interaction between the coefficients of the convection–diffusion equation produce schemes that are satisfactory in both accuracy and stability. The exponential finite difference scheme of the convective diffusion equation was first introduced by Allen and Southwell [9] to solve the second-order partial differential equation governing the transport of vorticity. The methods were analysed by Dennis [10,11] who found that the schemes were superior to the standard difference procedures and also developed several extensions of the methods. Spalding [12] and Roscoe [13] had also used different illustrations and independently developed similar exponential type schemes. Roscoe calls these exponential type schemes as unified difference schemes and used them for solving Navier–Stokes equations.

Dennis and Hudson [14] gave a basic formula of h^3 accuracy using an upwind approximation to a higher-order derivative and corrected it by adding a deferred correction to make it an h^4 scheme. These schemes were called as compact finite difference approximations. Finite difference schemes of exponential and polynomial orders for solving convection–diffusion problems were developed by Iyengar and Radhakrishna Pillai [15]. Compact finite difference schemes of order h^4 for the two-dimensional convection–diffusion type equations with constant and variable convection coefficients were developed by Mackinnon and Johnson [16]. By using the original model equation to represent the truncation terms, higher-order derivatives were replaced with lower-order derivatives and then approximated on a compact stencil. A perturbational h^4 compact exponential finite difference scheme was developed by Chen *et al.* [17] for the convective diffusion equation and numerical examples including one- to three-dimensional problems were solved to illustrate the behaviour of the developed exponential schemes.

The objective of the present study is to develop exponential type finite difference schemes to accurately represent convective diffusion equations. In ordinary differential equations, this includes two-point boundary value problems and second-order singular perturbation problems. In partial differential equations, the well-known Burgers equation, Navier–Stokes equations and elliptic singular perturbation problems fall into this category. In this study, only steady state problems are considered. Exponential type schemes are developed to solve linear and non-linear problems in one and two dimensions, in particular, fluid flow problems with boundary layers. Exponential schemes with non-uniform mesh are also derived to solve some specific problems. Comparisons with previously published results are given and the h^4 accuracy of the schemes is also computationally demonstrated using double precision arithmetic.

2. BASIC PROBLEM

To describe the basic approach, consider the steady, convection–diffusion model problem

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial u}{\partial x}=0 \quad 0<x<1, \quad u(0)=0, \quad u(1)=1 \tag{1}$$

where a and b are constants. Here a is the conductivity, b is the convective velocity and the solution u may represent heat, vorticity, etc. The exact solution to (1) is

$$u(x)=(\mathrm{e}^{sx}-1)/(\mathrm{e}^{s}-1) \tag{2}$$

where $s=-b/a$. When s becomes very large and positive, the $u\sim x$ curve is nearly horizontal at zero with a steep rise as x approaches unity. When s is large and negative, the horizontal part lies near unity and the steep part near $x=0$. For the numerical evaluation of such problems with boundary layers, a uniform mesh is unsuitable as $u(x)$ has large gradients and thus changes rapidly over small intervals. It is well known that central difference approximation to Equation (1) yields oscillatory solution if the cell Reynolds number $|bh/a|>2$.

Earlier investigators have generally concluded that for a convection–diffusion problem, there is no 'best formulation' for a first-derivative or second-derivative expression in isolation: it may be the combination of the first and second derivatives depending on the particular differential equation and the relative magnitude of the coefficient terms. Exponential-type schemes or finite difference approximations that had coefficients involving the exponential function can be effective for solving the above mentioned problems. For particular problems with boundary layer characteristics, accurate representation can be achieved by using non-uniform meshes and the numerical scheme has to be accordingly applied.

3. DEVELOPMENT OF THE SCHEMES: ONE-DIMENSIONAL CASE

3.1. Uniform mesh

Consider the steady one-dimensional non-homogeneous model problem

$$au_{xx}+bu_x=f(x) \quad 0\le x\le 1 \tag{3}$$

with the necessary boundary conditions, where a and b are constants and $f(x)$ is a sufficiently smooth function of x. The domain [0, 1] is uniformly subdivided into n intervals with $x_i=ih$, $h=x_{i+1}-x_i$, $u_i=u(x_i)$, $f_i=f(x_i)$ and $i\in\{0, 1, 2, \ldots, n\}$. Consider a finite difference approximation for Equation (3) at a grid point x_i as

$$c_1 D_h^2 u_i + c_2 D_h u_i = f_i \tag{4}$$

where $D_h u_i = (u_{i+1} - u_{i-1})/2h$ and $D_h^2 u_i = (u_{i+1} - 2u_i + u_{i-1})/h^2$ are the central difference approximations for the first and second derivatives and c_1 and c_2 are parameters to be determined. Equation (4) can be written in the form

$$c_1 D_h^2 u_i + c_2 D_h u_i = (au_{xx} + bu_x)_i \tag{5}$$

Making this equation exact for 1, x and $e^{-(bx)/a}$ (Gartland [18]), c_1 and c_2 can be determined. Hence the finite difference scheme (4) can be written as

$$\frac{bh}{2}\coth\left(\frac{bh}{2a}\right) D_h^2 u_i + bD_h u_i = f_i \tag{6}$$

Equation (6) is an exponential scheme of second-order accuracy for the convective diffusion model problem (3). This scheme gives rise to a diagonally dominant tri-diagonal system. Scheme (6) is similar to the method developed by Mackinnon *et al.* [16] using the original model equation to represent the truncation terms. Equation (6) can also be written as

$$\frac{b}{h(1-s)}[su_{i-1} - (1+s)u_i + u_{i+1}] = f_i \tag{7}$$

where $s = e^{-(bh)/a}$, which is a variant of the scheme developed by Allen and Southwell [9], Roscoe [13] and Iyengar and Radhakrishna Pillai [15].

3.2. Non-uniform mesh

For problems with boundary layers or where the variable has large gradients and thus changes rapidly on small intervals, the usage of non-uniform mesh becomes an essential aspect for the accurate representation of the solutions.

Let the successive steps of a non-uniform mesh be $h_1, h_2, \ldots, h_i, \ldots$, so that $x_k = \Sigma_{j=1}^k h_j$. Consider the finite difference approximation for Equation (3) at a grid point x_i as

$$c_1 D_{nh}^2 u_i + c_2 D_{nh} u_i = f_i \tag{8}$$

where c_1 and c_2 are parameters to be determined and

$$D_{nh} u_i = \frac{h_i^2 u_{i+1} - (h_i^2 - h_{i+1}^2)u_i - h_{i+1}^2 u_{i-1}}{h_i h_{i+1}(h_i + h_{i+1})}$$

$$D_{nh}^2 u_i = \frac{2[h_i u_{i+1} - (h_i + h_{i+1})u_i + h_{i+1} u_{i-1}]}{h_i h_{i+1}(h_i + h_{i+1})} \tag{9}$$

Equation (8) can be rewritten as

$$c_1 D_{nh}^2 u_i + c_2 D_{nh} u_i = (au_{xx} + bu_x)_i \tag{10}$$

Making this scheme exact for 1, x and $e^{-(bx)/a}$, the values of c_1 and c_2 are obtained as

$$c_1 = -\frac{b}{2}\frac{h_i^2(\bar{s}_{i+1}-1) - h_{i+1}^2(s_i-1)}{h_i(\bar{s}_{i+1}-1) + h_{i+1}(s_i-1)}, \qquad c_2 = b \tag{11}$$

where $s_i = e^{(bh_i)/a}$ and $\bar{s}_{i+1} = e^{-(bh_{i+1})/a}$. For uniform mesh with $h_i = h_{i+1} = h$, $c_1 = (bh/2)\coth[(bh)/(2a)]$ and results in the scheme (6). Scheme (8) with (11) for the non-uniform mesh can also be expressed in another form as

$$\frac{b[(1-\bar{s}_{i+1})u_{i-1} - (s_i - \bar{s}_{i+1})u_i + (s_i - 1)u_{i+1}]}{h_i(\bar{s}_{i+1}-1) + h_{i+1}(s_i-1)} = f_i \tag{12}$$

where s_i and $\bar{s}_{i+1}$ are as defined earlier. This is a second-order accurate scheme that produces a diagonally dominant tri-diagonal system.

Scheme (12) can also be obtained by another approach. Consider the finite difference approximation for Equation (3) at a grid point x_i as

$$c_{-1}u_{i-1} + c_0 u_i + c_1 u_{i+1} = (au_{xx} + bu_x)_i \tag{13}$$

where c_{-1}, c_0 and c_1 are the parameters to be determined. Making Equation (13) exact for 1, x and $e^{-(bx)/a}$ with the non-uniform mesh, we obtain the above approximation (12).

3.3. Development of a three-point fourth-order method

Consider the finite difference approximation for Equation (3) at a grid point x_i as

$$\alpha D_h^2 u_i + bD_h u_i = c_1 f_{i-1} + c_2 f_i + c_3 f_{i+1} \tag{14}$$

where

$$\alpha = \frac{bh}{2}\coth\left(\frac{bh}{2a}\right) \tag{15}$$

D_h^2 and D_h are as defined earlier and c_1, c_2 and c_3 are the parameters to be determined. Equation (14) can be rewritten as

$$\alpha D_h^2 u_i + bD_h u_i = c_1(au_{xx} + bu_x)_{i-1} + c_2(au_{xx} + bu_x)_i + c_3(au_{xx} + bu_x)_{i+1} \tag{16}$$

Equation (16) is exact for 1 and $e^{-(bx)/a}$. Making it exact for x, x^2 and x^3, and solving the resulting equations we obtain the coefficients as

$$c_1 = \frac{1}{6} - \frac{(\alpha - a)(2a + bh)}{2(bh)^2}, \qquad c_2 = \frac{2}{3} + \frac{2a(\alpha - a)}{(bh)^2}, \qquad c_3 = \frac{1}{6} - \frac{(\alpha - a)(2a - bh)}{2(bh)^2} \tag{17}$$

Scheme (14) with (17) for solving the model problem (3) is of fourth-order accuracy and produces a diagonally dominant tri-diagonal system.

3.4. Variable–coefficient case

Consider the steady one-dimensional non-homogeneous model problem

$$au_{xx} + b(x)u_x = f(x) \quad 0 \le x \le 1 \tag{18}$$

with the necessary boundary conditions, where a is a constant and b and f vary spatially. This equation is consistent with singular perturbation problems.

Consider the finite difference scheme with the uniform mesh for Equation (18) at a grid point x_i as

$$\alpha_i D_h^2 u_i + b_i D_h u_i = c_1 f_{i-1} + c_2 f_i + c_3 f_{i+1} \tag{19}$$

where

$$\alpha_i = \frac{b_i h}{2} \coth\left(\frac{b_i h}{2a}\right), \qquad c_1 = \frac{1}{6} - \frac{(\alpha_i - a)(2a + b_i h)}{2(b_i h)^2}, \qquad c_2 = \frac{2}{3} + \frac{2a(\alpha_i - a)}{(b_i h)^2}$$

$$c_3 = \frac{1}{6} - \frac{(\alpha_i - a)(2a - b_i h)}{2(b_i h)^2} \tag{20}$$

The truncation error (TE) of scheme (19) is given by

$$\mathrm{TE} = (au_{xx} + bu_x - f)_i + \left\{\frac{bh^2}{12a}(au_{xxx} + bu_{xx} - f_x) + \frac{h^2}{12}(au_{xxxx} + bu_{xxx} - f_{xx})\right\}_i + O(h^4) \tag{21}$$

Hence, scheme (19) is only of second-order accuracy. This scheme can be made fourth-order accurate by adding to it the following approximation to the $O(h^2)$ term in (21)

$$\left[\frac{bh^2}{12a} b_x u_x + \frac{h^2}{12}(2b_x u_{xx} + b_{xx} u_x)\right]_i \approx \left[\frac{bh^2}{12a} D_h b D_h u + \frac{h^2}{12}(2D_h b D_h^2 u + D_h^2 b D_h u)\right]_i \tag{22}$$

Hence, a fourth-order accurate exponential type scheme for solving the differential equation (18) is given by

$$AD_h^2 u_i + BD_h u_i = c_1 f_{i-1} + c_2 f_i + c_3 f_{i+1} \tag{23}$$

where

$$A = \alpha_i + \frac{h^2}{6} D_h b_i, \qquad B = b_i + \frac{h^2}{12}\left(\frac{b_i}{a} D_h b_i + D_h^2 b_i\right) \tag{24}$$

and α_i, c_1, c_2 and c_3 are given by Equation (20).

When $b(x) = b$ is a constant, scheme (23) reduces to the diagonally dominant scheme (14). Scheme (23) was applied (Section 5) to a number of problems with boundary layer and also problems of singular perturbation type. The numerical solutions for these problems were obtained by applying the scheme and solving the resultant tri-diagonal systems. The scheme was also applied with large number of subdivisions with uniform and non-uniform meshes, which show the stability of the method. The fourth-order accuracy of the scheme is also demonstrated numerically (Problem 5). Very accurate results were obtained for all the problems even with coarse grid sizes and the values obtained are compared with previous published results.

4. EXTENSION OF THE SCHEMES TO TWO-DIMENSIONS

4.1. Variable coefficients case

Scheme (23), developed for the one-dimensional variable coefficient case, can be extended to the two-dimensional case. Consider the boundary value problem

$$au_{xx} + b(x, y)u_x + cu_{yy} + d(x, y)u_y = f(x, y) \tag{25}$$

where a and c are constants and b, d and f vary spatially. This equation is consistent with the two-dimensional Navier–Stokes equations for constant viscosity. Let the step-length in the x-direction be $h = 1/n$ and in the y-direction be $k = 1/m$, where n and m are the numbers of subdivisions in the x- and y-directions respectively. Extending scheme (23) to the two-dimensional case, the approximation for Equation (25) at a point (x_i, y_j) can be written as

$$\begin{aligned}&\left(\alpha_1 + \frac{h^2}{6} D_h b\right)_{ij} D_h^2 u_{ij} + \left[b + \frac{h^2}{12}\left(\frac{b}{a} D_h b + D_h^2 b\right)\right]_{ij} D_h u_{ij} + \left(\alpha_2 + \frac{k^2}{6} D_k d\right)_{ij} D_k^2 u_{ij} \\ &+ \left[d + \frac{k^2}{12}\left(\frac{d}{c} D_k d + D_k^2 d\right)\right]_{ij} D_k u_{ij} = c_1 f_{i-1,j} + c_2 f_{ij} + c_3 f_{i+1,j} + c_4 f_{i,j-1} + c_5 f_{i,j+1}\end{aligned} \tag{26}$$

where

$$\alpha_1 = \frac{bh}{2}\coth\left(\frac{bh}{2a}\right), \qquad \alpha_2 = \frac{dk}{2}\coth\left(\frac{dk}{2c}\right) \tag{27}$$

$$D_h u_{ij} = (u_{i+1,j} - u_{i-1,j})/(2h), \qquad D_k u_{ij} = (u_{i,j+1} - u_{i,j-1})/(2k)$$

D_h^2 and D_k^2 are the central difference operators for the second derivatives with respect to x and y respectively and c_1, c_2, c_3, c_4 and c_5 are the parameters to be determined.

Writing the Taylor series expansions of the left and right sides of (26), using the differential equation and adding to scheme (26) the difference approximation to the leading terms of the truncation error, we obtain the fourth order accurate scheme to (26) at (x_i, y_j) as

$$[AD_h^2 + BD_k^2 + CD_h + D_1D_k + ED_hD_k + GD_hD_k^2 + HD_h^2D_k + KD_h^2D_k^2]u_{ij} = F_{ij} \tag{28}$$

where

$$A = \alpha_1 + \frac{h^2}{6}D_hb, \qquad B = \alpha_2 + \frac{k^2}{6}D_kd$$

$$C = b + \frac{h^2}{12}\left(\frac{b}{a}D_hb + D_h^2b\right) + \frac{k^2}{12}\left(\frac{d}{c}D_kb + D_k^2b\right)$$

$$D_1 = d + \frac{h^2}{12}\left(\frac{b}{a}D_hd + D_h^2d\right) + \frac{k^2}{12}\left(\frac{d}{c}D_kd + D_k^2d\right)$$

$$E = \frac{bd}{12}\left(\frac{h^2}{a} + \frac{k^2}{c}\right) + \frac{1}{6}(h^2D_hd + k^2D_kb) \tag{29}$$

$$G = \frac{b}{12}\left(\frac{h^2c}{a} + k^2\right), \qquad H = \frac{d}{12}\left(\frac{k^2a}{c} + h^2\right), \qquad K = \frac{1}{12}(h^2c + k^2a)$$

$$\alpha_1 = \frac{bh}{2}\coth\left(\frac{bh}{2a}\right), \qquad \alpha_2 = \frac{dk}{2}\coth\left(\frac{dk}{2c}\right)$$

$$F_{ij} = c_1f_{i-1,j} + c_2f_{ij} + c_3f_{i+1,j} + c_4f_{i,j-1} + c_5f_{i,j+1}$$

$$c_1 = \frac{1}{6} - \frac{(\alpha_1 - a)(2a + bh)}{2(bh)^2}, \qquad c_2 = \frac{1}{3} + \frac{2a(\alpha_1 - a)}{(bh)^2} + \frac{2c(\alpha_2 - c)}{(dk)^2}$$

$$c_3 = \frac{1}{6} - \frac{(\alpha_1 - a)(2a - bh)}{2(bh)^2}, \qquad c_4 = \frac{1}{6} - \frac{(\alpha_2 - c)(2c + dk)}{2(dk)^2}$$

$$c_5 = \frac{1}{6} - \frac{(\alpha_2 - c)(2c - dk)}{2(dk)^2}$$

Scheme (28) uses a nine-point stencil. Even though nine points are used, the successive line overrelaxation (SLOR) approach with an alternating direction implicit procedure enables us to obtain the solutions of the problems by solving tri-diagonal systems. Accurate results were obtained for several problems including a two-dimensional Navier–Stokes-type problem which show the stability of the scheme. The fourth-order accuracy of scheme (28) is also illustrated numerically (Problem 7). A variant to this scheme was developed by Mackinnon and Johnson [16] using a different approach. Explicit stability analysis of this scheme appears to be quite complex (Dekema and Schultz [6], Mackinnon and Johnson [16]).

4.2. Constant coefficient case

Consider the boundary value problem

$$au_{xx} + bu_x + cu_{yy} + du_y = f(x, y) \tag{30}$$

where a, b, c and d are constants. In this case, scheme (28) simplifies as

$$[AD_h^2 + BD_k^2 + CD_h + D_1D_k + ED_hD_k + GD_hD_k^2 + HD_h^2D_k + KD_h^2D_k^2]u_{ij} = F_{ij} \tag{31}$$

where $A = \alpha_1,\ B = \alpha_2,\ C = b,\ D_1 = d$

$$E = \frac{bd}{12}\left(\frac{h^2}{a} + \frac{k^2}{c}\right)$$

$$G = \frac{b}{12}\left(\frac{h^2c}{a} + k^2\right), \qquad H = \frac{d}{12}\left(\frac{k^2a}{c} + h^2\right), \qquad K = \frac{1}{12}(h^2c + k^2a) \tag{32}$$

$$F_{ij} = c_1f_{i-1,j} + c_2f_{ij} + c_3f_{i+1,j} + c_4f_{i,j-1} + c_5f_{i,j+1}$$

and α_1, α_2, c_1, c_2, c_3, c_4 and c_5 have the same form as in (29), but are constants. This scheme is of fourth-order accuracy and uses a nine-point stencil.

5. COMPUTATIONAL RESULTS

The schemes developed in the previous sections are applied on a number of linear and non-linear problems, which include boundary layer problems, elliptic singular perturbation problems and a two-dimensional Navier–Stokes equation. For linear one-dimensional problems, a tri-diagonal system is solved and for non-linear problems, an iterative successive overrelaxation procedure is associated with the tri-diagonal system solution. For two-dimensional problems, the schemes are applied using a SLOR procedure along with an alternating direction, which enables us to deal with tri-diagonal systems.

The iterative procedure is repeated until $|u^{(k+1)} - u^{(k)}| \leq \delta$ for all grid points, where k is the iterative count and $\delta = 10^{-10}$ is the selected tolerance. Results are compared with the corresponding exact solutions or previously published results. The fourth-order accuracy of the schemes in one and two-dimensions are also demonstrated numerically for the variable coefficient cases.

5.1. One-dimensional constant coefficient problems

5.1.1. Problem 1 (Dekema and Schultz [6] and Segal [19]).

$$-\varepsilon u_{xx}+u_x=0, \quad u(0)=0, \quad u(1)=1.0 \tag{33}$$

The exact solution is given by $u(x)=(e^{x/\varepsilon}-1)/(e^{1/\varepsilon}-1)$.

A detailed analysis of the problem is given in Dekema and Schultz [6] and Segal [19] and they have stated that as ε was decreased it was necessary to use a divided interval to obtain accurate results. The left side of the interval $[0, 1-8\varepsilon]$, was divided into n subdivisions each of length $h_1=(1-8\varepsilon)/n$ and the right side of the interval $[1-8\varepsilon, 1]$, was divided into m subdivisions, each of length $h_2=8\varepsilon/m$. They found that better results could be obtained by using the subregions [0, 0.999] and [0.999, 1.0].

Using the fourth-order accurate scheme (14) for constant coefficients, accurate results were obtained even for very small ε without going for the divided interval. For comparison purpose, results were also obtained with divided intervals and scheme (12) was used to approximate the equation at the dividing point. We note that the Dekema and Schultz [6] method is a $O(h^{10})$ method. The results are given in Table I. Some of the results in Table I are with the divided intervals $[0, 1-8\varepsilon]$ and $[1-8\varepsilon, 1]$ where n and m are the corresponding subdivisions as given earlier. In Table I, for a fixed number of nodes, $n=1000$, the maximum absolute error goes from 0.33×10^{-15} to 0.31×10^{-16} when ε goes from 10^{-3} to 10^{-4}. Results of Dekema and Schultz [6] also show a similar behaviour. A similar behaviour was also noticed in some other cases, especially when the maximum absolute error was very small. This behaviour may be due to the fact that the range of the error is too small for a representation in double precision.

Results obtained by dividing the region into the two subregions [0, 0.999] and [0.999, 1.0] are given in Table II. In Table II, the error for $\varepsilon=10^{-5}$, does not decrease when the number of nodes increases. A similar behaviour is also seen with the results of Dekema and Schultz [6]. This type of behaviour was also noticed in some other cases, particularly when the maximum absolute error was very small. Again, it may be because the range of the error is very small for a representation in double precision.

5.1.2. Problem 2 (Dekema et al. [6] and Segal [19]).

$$-\varepsilon u_{xx}+u_x=\varepsilon\pi^2\sin\pi x+\pi\cos\pi x, \quad u(0)=0, \quad u(1)=1.0 \tag{34}$$

The exact solution is given by $u(x)=\sin\pi x+(e^{x/\varepsilon}-1)/(e^{1/\varepsilon}-1)$.

This problem was analysed by Dekema and Schultz [6] in detail and they have obtained the results with their $O(h^4)$, $O(h^6)$, $O(h^8)$ and $O(h^{10})$ methods. They have stated that for all the methods with $\varepsilon=10^{-5}$, it was necessary to use a very small overrelaxation factor (0.0125) and initial values from the $\varepsilon=10^{-4}$ run, in order to obtain a converging solution. They also found that better results could be obtained if they did not divide the region into the two subregions $[0, 1-8\varepsilon]$ and $[1-8\varepsilon, 1]$, but instead used the subregions [0, 0.999] and [0.999, 1.0].

We have obtained accurate results using the fourth-order scheme (14) which is even comparable with the results obtained by the $O(h^{10})$ method. For $\varepsilon=10^{-5}$ and 10^{-6}, very accurate results were obtained by our scheme even without resorting to the divided interval.

Table I. Maximum absolute error for fourth-order scheme (14) and for tenth-order method of Dekema *et al.* [6]. Problem 1.

$\varepsilon = 10^{-3}$			$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-5}$		
n or (n, m)	Scheme (14)	Dekema's $O(h^{10})$ method	n or (n, m)	Scheme (14)	Dekema's $O(h^{10})$ method	n or (n, m)	Scheme (14)	Dekema's $O(h^{10})$ method
500	0.25×10^{-15}	0.5×10^{-5}	1000	0.31×10^{-16}	0.7×10^{-8}	10 000	0.54×10^{-16}	[a]
1000	0.33×10^{-15}	0.8×10^{-8}	10 000	0.41×10^{-13}	[a]	3000, 100	0.55×10^{-11}	0.4×10^{-5}
2000	0.33×10^{-13}	0.7×10^{-11}	4000, 100	0.31×10^{-12}	0.2×10^{-6}	10 000, 100	0.40×10^{-11}	[a]
1000, 100	0.42×10^{-13}	0.15×10^{-10}	5000, 100	0.11×10^{-11}	0.2×10^{-7}			

[a] Not available.

Table II. Maximum absolute error for fourth-order scheme (14) and for $O(h^{10})$ method of Dekema *et al.* [6] using the sub-regions [0, 0.999] and [0.999, 1.0]. Problem 1.

$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-5}$			$\varepsilon = 10^{-6}$		
(n, m)	Scheme (14)	Dekema's $O(h^{10})$ method	(n, m)	Scheme (14)	Dekema's $O(h^{10})$ method	(n, m)	Scheme (14)	Dekema's $O(h^{10})$ method
(1000, 40)	0.12×10^{-11}	0.5×10^{-1}	(1000, 1000)	0.14×10^{-10}	0.7×10^{-10}			
(4000, 100)	0.10×10^{-11}	0.3×10^{-7}	(4000, 100)	0.31×10^{-11}	0.8×10^{-8}			
(4000, 1000)	0.15×10^{-11}	0.3×10^{-7}	(4000, 1000)	0.14×10^{-10}	0.8×10^{-10}	(4000, 1000)	0.56×10^{-10}	0.7×10^{-8}

Table III. Maximum absolute error for fourth-order exponential scheme (14) and for $O(h^{10})$ method of Dekema *et al.* [6]. Problem 2.

$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-5}$			$\varepsilon = 10^{-6}$		
n or (n, m)	Scheme (14)	Dekema's $O(h^{10})$ method	n or (n, m)	Scheme (14)	Dekema's $O(h^{10})$ method	n or (n, m)	Scheme (14)	Dekema's $O(h^{10})$ method
2000	0.29672×10^{-7}	0.6×10^{-2}	3000, 10	0.29740×10^{-7}	[a]	50 000	0.29801×10^{-7}	[a]
5000	0.29795×10^{-7}	0.6×10^{-5}	25 000	0.29801×10^{-7}	0.95×10^{-2}	5000, 100	0.29783×10^{-7}	[a]
7000	0.29799×10^{-7}	0.2×10^{-6}	45 000	0.29802×10^{-7}	0.7×10^{-3}			
10 000	0.29801×10^{-7}	0.7×10^{-8}	50 000	0.29804×10^{-7}	[a]			
3000, 10	0.29768×10^{-7}	0.5×10^{-5}						
4000, 10	0.29789×10^{-7}	0.2×10^{-6}						

[a] Not available.

Table IV. Maximum absolute error for fourth-order exponential scheme and for $O(h^{10})$ method of Dekema *et al.* [6] using the subregions [0, 0.999] and [0.999, 1.0]. Problem 2.

$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-5}$			$\varepsilon = 10^{-6}$		
n, m	Scheme (14)	Dekema's $O(h^{10})$ method	n, m	Scheme (14)	Dekema's $O(h^{10})$ method	n, m	Scheme (14)	Dekema's $O(h^{10})$ method
4000, 100	0.29789×10^{-7}	0.3×10^{-7}	400, 500	0.21666×10^{-6}	0.2×10^{-3}	4000, 1000	0.29771×10^{-7}	0.7×10^{-3}
9000, 1000	0.29801×10^{-7}	0.8×10^{-11}	500, 50	0.12308×10^{-6}	0.3×10^{-4}	8000, 2000	0.29794×10^{-7}	0.3×10^{-5}
			1000, 100	0.28498×10^{-7}	0.6×10^{-7}	9000, 1000	0.29795×10^{-7}	0.1×10^{-6}

Results are presented in Table III. Some of the results are obtained by using the subregions $[0, 1-8\varepsilon]$ and $[1-8\varepsilon, 1.0]$, where n and m are the corresponding number of subdivisions. Table IV shows similar results using the subregions [0, 0.999] and [0.999, 1.0]. It was seen that the new schemes consistently gave accurate results even for very small values of ε. Maximum absolute errors obtained by using the subregions [0, 0.999] and [0.999, 1.0] with $n=4000$ and $m=1000$ for values of ε up to 10^{-12} were in the order of 10^{-7}.

5.1.3. Problem 3 (White [20]).

$$\varepsilon u_{xx} + u_x - \varepsilon u = -\sin x, \quad 0 < x < \pi, \text{ with } u(0) = 0, \quad u(\pi) = 0 \tag{35}$$

The exact solution of the problem is given by

$$u(x) = \frac{1}{1+4\varepsilon^2}\left(A\ e^{(r_1 x)/\pi} + B\ e^{(r_2 x)/\pi} + 2\varepsilon \sin x + \cos x\right) \tag{36}$$

where $r_1 = -\pi(1+\sqrt{1+4\varepsilon^2})/(2\varepsilon)$, $r_2 = 2\pi\varepsilon(1+\sqrt{1+4\varepsilon^2})^{-1}$, $A = (1+e^{r_2})/(e^{r_1}-e^{r_2})$, $B = (1+e^{r_1})/(e^{r_2}-e^{r_1})$.

The calculations were carried out for various values of ε with $n=100$ and 1000 using the fourth-order scheme (14). This problem has a boundary layer at $x=0$ and accurate results were obtained with the new exponential schemes. Maximum absolute errors obtained are given in Table V.

5.2. One-dimensional variable coefficient problems

5.2.1. Problem 4 (Gartland [18]).

$$\varepsilon u_{xx} - \frac{1}{1+x} u_x - \frac{1}{2+x} u = f(x) \quad 0 < x < 1 \tag{37}$$

$$u(0) = 1 + 2^{-1/\varepsilon}, \quad u(1) = e + 2 \tag{38}$$

and $f(x)$ is computed from the exact solution

$$u(x) = e^x + 2^{-(1/\varepsilon)}(1+x)^{1+(1/\varepsilon)} \tag{39}$$

Table V. Maximum absolute errors for the fourth-order exponential scheme (14). Problem 3.

ε	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}
$n=100$ ($h=0.03142$)	0.81832×10^{-8}	0.16499×10^{-4}	0.13068×10^{-5}	0.13072×10^{-5}	0.13072×10^{-5}
$n=1000$ ($h=0.003142$)	0.76432×10^{-8}	0.57727×10^{-7}	0.59612×10^{-7}	0.59631×10^{-7}	0.59632×10^{-7}

The problem was solved with $n = 256$, 512 and 1024 for various values of ε using scheme (23) for the variable coefficient case. Maximum absolute errors are given in Table VI.

5.2.2. Problem 5. This problem was considered by Chen *et al.* [17] for testing their perturbational h^4 exponential finite difference scheme. The model equation of fluid flow considered was the Burgers equation

$$u\frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} \quad 0 \leq x \leq 1 \tag{40}$$

with boundary conditions to give its solution as

$$u(x) = \tanh[Re(1-2x)/4] \tag{41}$$

For large values of the Reynolds number Re, the solution contains an abrupt change centred at the point $x = 0.5$. Initially, the calculations were carried out for $Re = 10$ and the computational region $0 \leq x \leq 1$ was distributed uniformly with 20 grid points ($h = 1/19$) as was done by Chen *et al.* [17]. The iterative procedure is repeated until the solution converges throughout the domain with the tolerance $\delta = 10^{-10}$. Solutions obtained using the fourth-order exponential scheme (23) and the $O(h^4)$ schemes of Dennis and Hudson [14] and Chen *et al.* [17] are given in Table VII. To verify the $O(h^4)$ accuracy of the present scheme, computations were also carried out using $h = 1/38$. The ratio of the errors is approximately 16, confirming the $O(h^4)$ accuracy of the scheme. In addition to this, $\log(E)$ versus $\log(h)$ was plotted with $h = 1/19$ and 1/38, where $E = |u(x_i) - u_i|$. The slope of the straight line was evaluated and these values are approximately 4, again confirming the $O(h^4)$ accuracy of the scheme. These slope values are also given in Table VII.

For convection dominated cases with large Re, steep changes take place near the centre point $x = 0.5$. To resolve these steep changes and also for a more accurate representation, the given interval [0, 1] was subdivided into the three intervals $[0, 0.5 - 20\varepsilon]$, $[0.5 - 20\varepsilon, 0.5 + 20\varepsilon]$, $[0.5 + 20\varepsilon, 1.0]$ where $\varepsilon = 1/Re$. The number of subdivisions for the corresponding subintervals were selected as 10, 100 and 10 respectively. Because of the non-uniform mesh involved, scheme (12) was used at the dividing point while scheme (23) was used at the other points. Even though the h^2 accurate scheme was used at the dividing point along with the h^4 accurate scheme for other points, accurate results were obtained. Using the non-uniform mesh, accurate and smooth solutions were obtained with $Re = 500$ and 10^5 and the maximum absolute errors

Table VI. Maximum absolute errors for the fourth-order exponential scheme (23). Problem 4.

ε	$n = 256$	512	1024
1.0	0.20541×10^{-8}	0.20529×10^{-8}	0.20554×10^{-8}
10^{-1}	0.17769×10^{-7}	0.17876×10^{-7}	0.17882×10^{-7}
10^{-2}	0.59980×10^{-7}	0.29857×10^{-7}	0.34610×10^{-7}
10^{-3}	0.73234×10^{-4}	0.52059×10^{-5}	0.30735×10^{-6}

Table VII. Solutions of the Burgers equation with $Re = 10$. Problem 5.

x	Exact	U ($h = 1/19$) [14]	U ($h = 1/19$) [17]	U_D ($h = 1/19$) Scheme (23)	U_D^* ($h = 1/38$) Scheme (23)	$\frac{\text{Error in } U_D}{\text{Error in } U_D^*}$	Slope
0.3158	0.72639501	0.7264	0.7264	0.72638599	0.72639442	15.46	3.95
0.3684	0.57696066	0.5769	0.5769	0.57696434	0.57696085	19.74	4.30
0.4211	0.37543661	0.3754	0.3754	0.37545164	0.37543749	17.22	4.11
0.4737	0.13082483	0.1308	0.1308	0.13083398	0.13082536	17.11	4.10
0.5263	−0.13082483	−0.1318	−0.1308	−0.13083398	−0.13082536	17.11	4.10
0.5789	−0.37543661	−0.3754	−0.3754	−0.37545164	−0.37543749	17.22	4.11
0.6316	−0.57696066	−0.5769	−0.5769	−0.57696434	−0.57696085	19.74	4.30
0.6842	−0.72639501	−0.7264	−0.7264	−0.7238599	−0.72639442	15.46	3.95

were 0.88753×10^{-4} and 0.88787×10^{-4} respectively. The solutions near the central point 0.5 are presented in Figures 1 and 2.

5.3. Two-dimensional case

5.3.1. Problem 6. Consider the differential equation

$$u_{xx} + u_{yy} + \omega u_x = 0 \tag{42}$$

with the boundary conditions

$$u(0, y) = y(1-y), \qquad u(1, y) = \left(y(1-y) - \frac{2}{\omega}\right) e^{-\omega}$$

and

$$u(x, 0) = -\frac{2x}{\omega} e^{-\omega x} = u(x, 1)$$

The exact solution is

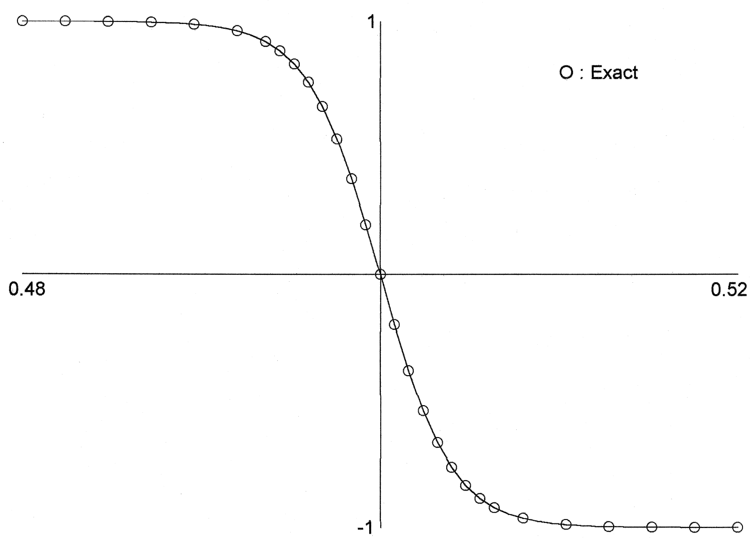

Figure 1. Solutions near the centre point for $Re = 500$, Problem 5. Total number of nodal points in $[0, 1] : 121$; $[0.48, 0.52] : 51$ (non-uniform mesh).

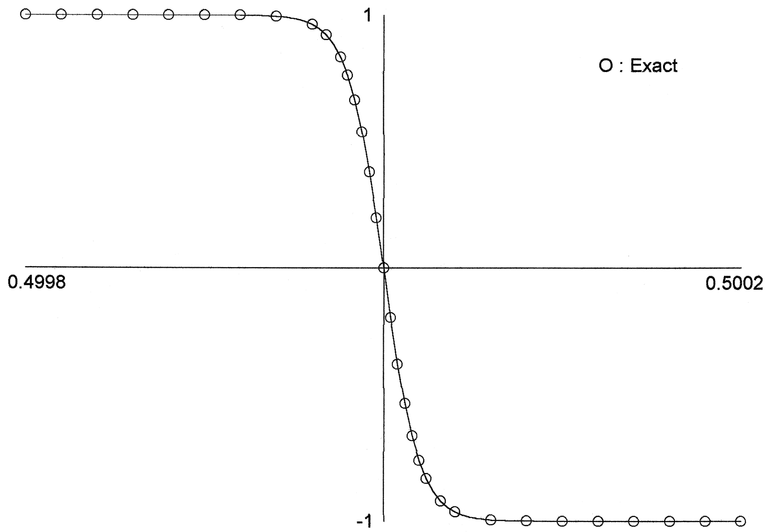

Figure 2. Solutions near the centre point for $Re = 10^5$, Problem 5. Total number of nodal points in [0, 1] : 121; [0.4998, 0.5002] : 101 (non-uniform mesh).

$$u(x, y) = \left(y(1-y) - \frac{2x}{\omega} \right) \mathrm{e}^{-\omega x} \tag{43}$$

This problem has a boundary layer. Scheme (31) was used for solving this problem with n and m as the number of subdivisions in the x- and y-directions respectively. SLOR with an alternating direction procedure was used to obtain the solution of the system of equations. Even though scheme (31) uses a nine-point stencil, this approach enables us to deal with tri-diagonal systems. The method is stable and converged rapidly for all values of ω attempted. Results are presented in Table VIII.

Table VIII. Maximum absolute errors for scheme (31). Problem 6.

(n, m)	ω					
	10	40	10^2	10^3	10^4	10^5
(10, 10)	0.10×10^{-3}	0.86×10^{-3}	0.13×10^{-2}	0.18×10^{-2}	0.18×10^{-2}	0.18×10^{-2}
(10, 20)	0.65×10^{-4}	0.54×10^{-3}	0.84×10^{-3}	0.12×10^{-2}	0.13×10^{-2}	0.13×10^{-2}
(20, 10)	0.16×10^{-4}	0.24×10^{-3}	0.64×10^{-3}	0.10×10^{-2}	0.11×10^{-2}	0.11×10^{-2}
(20, 20)	0.62×10^{-5}	0.97×10^{-4}	0.25×10^{-3}	0.43×10^{-3}	0.45×10^{-3}	0.46×10^{-3}
(40, 20)	0.96×10^{-6}	0.17×10^{-4}	0.83×10^{-4}	0.25×10^{-3}	0.27×10^{-3}	0.27×10^{-3}
(40, 40)	0.38×10^{-6}	0.68×10^{-5}	0.33×10^{-4}	0.10×10^{-3}	0.11×10^{-3}	0.11×10^{-3}

5.3.2. Problem 7. Consider the following two-dimensional model equations for fluid flow

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}=(2\sin y+\sin x)\cos x$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-u\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}=(\sin y-2\sin x)\cos y \quad 0\leq x,\ y\leq \pi \tag{44}$$

which have the exact solution

$$u(x,y)=-\cos x\sin y, \qquad v(x,y)=\sin x\cos y \tag{45}$$

This problem is a modification of the one considered by Roscoe [13] as a test problem for the developed methods which were similar to those of Allen and Southwell [9]. Dennis *et al.* [11,14] used this problem to demonstrate and verify the h^4 accurate method developed by them. To illustrate the h^4 accurate perturbational exponential scheme, Chen *et al.* [17] simplified this problem by substituting the exact value of v in (44) and then considered only the first equation for u. The terms on the right-hand side of (44) do not themselves constitute possible pressure gradients to be truly representative of the two-dimensional Navier–Stokes equations, but (45) satisfies the equation of continuity and the differential equations (44). Thus the essential features of a model solution of the Navier–Stokes equations are preserved by this problem.

Equations (44) have been solved using the exponential scheme (28) with the same step-length $h=\pi/20$ in both the x- and y-directions. The iterative procedure consists of an overall sequence of iterations between the two equations in (44) which itself is composed of a sequence of a SLOR approach with an alternating direction implicit procedure for each equation. The procedure was started with the initial values of u and v taken as the interpolated values from the boundaries, and the iterative procedure was repeated until u and v converge throughout the solution domain. For convergence, a tolerance $\delta=10^{-10}$ was taken. The iterative procedure converged rapidly. The results for $u(0.7\pi, y)$ for values of y/π from 0.1 to 0.5 are given in Table IX. The solutions obtained by the fourth-order accurate schemes of Dennis *et al.* [14] and Chen *et al.* [17] are also given in Table IX. As mentioned earlier, Chen *et al.* [17] obtained solutions for u by solving only the first equation in (44) as the exact values of v were used. It

Table IX. Solutions of Problem 7 with $h=\pi/20$ for $u(0.7\pi, y)$, $y/\pi=0.1$–0.5.

y/π	Exact solution	Dennis *et al.* [14]	Chen *et al.* [17]	Solution with scheme (28)
0.1	0.1816356	0.1816368	0.1816368	0.1816368
0.2	0.3454915	0.3454932	0.3454934	0.3454931
0.3	0.4755283	0.4755299	0.4755308	0.4755299
0.4	0.5590170	0.5590185	0.5590201	0.5590185
0.5	0.5877852	0.5877867	0.5877886	0.5877868

Table X. Order of the present scheme (28). Problem 7.

y/π	Exact	$h=\pi/10$ (U_B)	$h=\pi/20$ (U_B^*)	$\frac{\text{Error in } U_B}{\text{Error in } U_B^*}$	Slope
0.1	0.18163563	0.18165439	0.18163678	16.35	4.03
0.2	0.34549150	0.34551787	0.34549311	16.45	4.04
0.3	0.47552827	0.47555581	0.47552991	16.61	4.05
0.4	0.55901700	0.55904335	0.55901855	16.78	4.07
0.5	0.58778524	0.58781087	0.58778679	16.86	4.07

is seen that accurate results are obtained with the new exponential type schemes even with coarse mesh.

Computations were also performed with step-length $h=\pi/10$. The ratio of the errors in solutions using $h=\pi/10$ and $\pi/20$ is found to be approximately 16, confirming the fourth-order accuracy of the scheme (Chen *et al.* [17]). The log(E) versus log(h) was also plotted with $h=\pi/10$ and $\pi/20$, where $E=|u(x_i, y_j)-u_{ij}|$. The slope of the straight line was evaluated and these values were found to be close to 4, again confirming the $O(h^4)$ accuracy of the scheme. These results for $u(0.7\pi, y)$ for values of y/π from 0.1 to 0.5 are given in Table X.

6. CONCLUSIONS

In this paper, fourth-order accurate finite difference methods of an exponential nature are presented for solving one- and two-dimensional convection–diffusion type differential equations with constant and variable coefficients. In the one-dimensional case, the approximating equation is developed using three points, which results in a tri-diagonal system. For the two-dimensional problems, the schemes use a nine-point stencil. However, the SLOR approach with an alternating direction implicit procedure enables us to deal with only tri-diagonal systems. The schemes were also applied with non-uniform mesh to resolve the solution in regions of steep variations. For this implementation, an exponential finite difference scheme on non-uniform mesh is derived to approximate the differential equation at the dividing point. By this approach, better results were obtained for particular problems even with less numbers of nodal points.

The new schemes were applied to a number of linear and non-linear problems in one and two dimensions. In particular, these schemes are very useful in the solution of fluid flow problems. In all the problems, accurate and smooth solutions (without any oscillations) were obtained. Most of the results obtained were compared with the earlier published results. The h^4 accuracy of the schemes in both one and two dimensions was also illustrated numerically. The effectiveness of the schemes developed has been demonstrated in terms of accuracy and rate of convergence for seven test problems.

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